DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

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AN OVERLAPPING GENERATIONS MODEL CORE EQUIVALENCE THEOREM

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SOCIAL SCIENCE WORKING PAPER 706

September 1989

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ABSTRACT

The classical Debreu–Scarf core equivalence theorem asserts that in an exchange economy with a finite number of agents an allocation (under certain conditions) is a Walrasian equilibrium if and only if it belongs to the core of every replica of the exchange economy. The pioneering work of P. Samuelson has shown that such a result fails to be true in exchange economies with a countable number of agents.

This paper presents a Debreu–Scarf type core equivalence theorem for the overlapping generations (OLG) model. Specifically, the notion of a short-term core allocation for the overlapping generations model is introduced and it is shown that (under some appropriate conditions) an OLG model allocation is a Walrasian equilibrium if and only if it belongs to the short-term core of every replica of the OLG economy.

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We show that in the overlapping generations model an allocation is a Walrasian equilibrium if and only if the allocation belongs to the short-term core of every replication of the economy—and hence, we establish a core equivalence theorem for the overlapping generations model.

1. INTRODUCTION

In their classic paper G. Debreu and H. E. Scarf [13] established that an allocation in an exchange economy with a finite number of agents and a finite dimensional commodity space is a Walrasian equilibrium if and only if it belongs to the core of every n-fold replica of the economy. The Debreu–Scarf theorem was extended to exchange economies and economies with production having a finite number of agents and infinitely many commodities by C. D. Aliprantis, D. J. Brown and O. Burkinshaw in [1] and [2]. However, the Debreu–Scarf theorem fails to be true in exchange economies with a countable number of agents, as was first observed by P. Samuelson in his pioneering paper on the Consumption Loan Model [19]. For in the overlapping generations (OLG) model not every competitive equilibrium is Pareto optimal. In this work, we search for the analogue of the Debreu–Scarf theorem in the overlapping generations model. We establish that in the overlapping generations model an allocation is a Walrasian equilibrium if and only if it belongs to the short-term core of every replica of the overlapping generations model. This result is new even for overlapping generations models with finite dimensional commodity spaces.

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* We thank D. J. Brown and an anonymous referee for their constructive comments on an earlier version of this paper. Research of both authors was supported in part by a Chrysler Corporation grant to IUPUI. The article in its present form will appear in the *Journal of Economic Theory*.

For economies with a finite dimensional commodity space, versions of the first and second welfare theorems were established by G. Debreu in [12] and were extended to economies with an infinite dimensional commodity space by A. Mas-Colell [18], C. D. Aliprantis and O. Burkinshaw [8] and M. A. Khan and R. Vohra [15]. The first and second welfare theorems dealing with Pareto optimality in the overlapping generations model were studied by Y. Balasko and K. Shell [9, 10] under the assumption that agents' consumption sets lie in a finite dimensional space and their preferences are given by smooth functions. Balasko and Shell were able to prove analogues of both the first and second welfare theorems by using a notion of optimality which is weaker than Pareto optimality. We shall refer to this optimality concept as short-term optimality. Short-term optimality was studied by C. D. Aliprantis, D. J. Brown and O. Burkinshaw in [4]; the monograph [5] contains more about this optimality notion. As mentioned above, we shall also introduce a core notion for the overlapping generations model which we shall refer to as the short-term core of the overlapping generations model. J. Esteban in [14] characterized the Walrasian equilibria that are core allocations and gave an example where the sets of Walrasian equilibria and core allocations are not equal. In an OLG model with a continuum of agents, S. Chae [11] showed that an allocation is a Walrasian equilibrium exactly when it cannot be improved upon by any coalition in the short-term and thus he established a core equivalence theorem for a continuum of agents.

In sum: We address the relationship between the welfare and market mechanisms in the overlapping generations model with an infinite dimensional space of commodities. In particular, we are interested in what sense competitive equilibria in overlapping generations models over an infinite dimensional commodity space are related to the core of the economy—in short, we are interested in formulating a core equivalence theorem for the overlapping generations model in analogy with the Debreu–Scarf theorem. The main results of the paper are Theorems 4.2 and 5.1 which characterize the Walrasian equilibria in an overlapping generations model in terms of the short-term core.

2. MATHEMATICAL PRELIMINARIES

This paper will utilize the theory of Riesz spaces. For detail accounts of the theory of Riesz spaces see the books [6, 7, 16, 20, 23]. A few basic facts about Riesz spaces are briefly discussed below.

A Riesz space is a partially ordered (real) vector space which is in addition a lattice, i.e., with the extra property that finite sets have suprema (least upper bounds) and infima (greatest lower bounds). The supremum and infimum of two elements x and y will be denoted by $x \vee y$ and $x \wedge y$, respectively. That is,

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}.$$

If x is an element in a Riesz space, then the elements

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x),$$

are referred to as the positive part, the negative part and the absolute value of x, respectively.

For the rest of the discussion in this section the letter E will denote a Riesz space. The positive cone of E will be denoted by E^+, i.e.,

$$E^+ = \{x \in E\colon x \geq 0\}.$$

A vector subspace A of E is said to be an *ideal* whenever $|y| \leq |x|$ and $x \in A$ imply $y \in A$. Every element x belongs to a smallest ideal A_x, called the *principal ideal generated by* x. We have

$$A_x = \{y \in E\colon \text{There exists } \lambda > 0 \text{ with } |y| \leq \lambda |x|\}.$$

The $\|\cdot\|_\infty$-norm on A_x is the lattice norm defined by

$$\|y\|_\infty = \inf\{\lambda > 0\colon |y| \leq \lambda |x|\}, \quad y \in A_x.$$

Under the $\|\cdot\|_\infty$-norm, A_x is a normed lattice (in fact, an M-space with unit). The norm dual of $(A_x, \|\cdot\|_\infty)$ will be designated by A'_x. More generally, every non-empty subset A of E is contained in a smallest ideal, called the ideal generated by A. The ideal A enjoys some remarkable algebraic and topological properties that will be employed in our study. For details regarding the properties of the ideal A we refer the reader to [3].

The sets of the form $[x, y] := \{z \in E\colon x \leq z \leq y\}$, where $x \leq y$, are called the order intervals of E. A linear functional $f\colon E \longrightarrow \mathcal{R}$ is said to be *order bounded* whenever f carries order intervals onto bounded subsets $\mathcal{R}$. The vector space of all order bounded linear functionals on E is called the *order dual* of E and is denoted by $E^\sim$. Under the ordering $f \geq g$ whenever $f(x) \geq g(x)$ for each $x \in E^+$, the order dual $E^\sim$ is a Dedekind complete Riesz space.

The commodity-price duality in our economic model will be described by a Riesz dual system. A *Riesz dual system* $\langle E, E' \rangle$ is a Riesz space E together with an ideal E' of $E^\sim$ that separates the points of E such that the duality is the natural one, i.e., $\langle x, x' \rangle = x'(x)$ holds for all $x \in E$ and all $x' \in E'$. We will assume that each Riesz space E is equipped with a locally convex-solid topology that is consistent with the dual system $\langle E, E' \rangle$; see [6, 7].

3. THE OVERLAPPING GENERATIONS MODEL

In the overlapping generations model the index t will denote the time period. The commodity-price duality at period t will be represented by a Riesz dual system $\langle E_t, E_t' \rangle$. Consequently, we have a sequence $(\langle E_1, E_1' \rangle, \langle E_2, E_2' \rangle, \ldots)$ of Riesz dual systems each member of which designates the commodity-price duality at the corresponding time period. Note that we allow a (possibly) different commodity space at each time period in order to accommodate the possibility that future commodities may enter the market and old ones may disappear.

It will be assumed that each consumer has a two-period lifetime. Thus, consumer t is born at period t and lives all his life in periods t and $t+1$. Each consumer trades and has tastes for commodities only during his life-time period. We suppose that consumer t gets an initial endowment $0 < \omega_t^t \in E_t$ at period t and $0 < \omega_t^{t+1} \in E_{t+1}$ at period $t+1$ (and, of course, nothing else in any other periods). Consequently, his initial endowment ω_t can be represented by the vector

$$\omega_t = (0, \ldots, 0, \omega_t^t, \omega_t^{t+1}, 0, 0, \ldots)$$

where ω_t^t and ω_t^{t+1} occupy the positions t and $t+1$, respectively. Also, we shall assume that the "father" of consumer 1 (i.e., consumer 0) is present in the model at period 1. He will be designated as consumer 0 and his endowment will be taken to be of the form

$$\omega_0 = (\omega_0^1, 0, 0, \ldots)$$

with $0 < \omega_0^1 \in E_1$.

The vectors of the form

$$\mathbf{x}_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, 0, \ldots),$$

where $x_t^t \in E_t^+$ and $x_t^{t+1} \in E_{t+1}^+$ represent the commodity bundles for consumer t during his life time. Each consumer t maximizes a utility function u_t defined on his commodity space, i.e., u_t is a function from $E_t^+ \times E_{t+1}^+$ into $\mathcal{R}$. The value of u_t at the commodity bundle $\mathbf{x}_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, 0, \ldots)$ will be denoted by $u_t(x_t^t, x_t^{t+1})$.

The utility functions will be assumed to satisfy the following properties.

1. *Each u_t is quasi-concave;*
2. *Each u_t is strictly monotone on $E_t^+ \times E_{t+1}^+$, that is, $(x, y) > (x_1, y_1)$ in $E_t^+ \times E_{t+1}^+$ implies $u_t(x, y) > u_t(x_1, y_1)$; and*
3. *Each u_t is continuous on $E_t^+ \times E_{t+1}^+$, where each E_t is assumed equipped with a locally convex-solid topology consistent with the duality $\langle E_t, E_t' \rangle$.*

The case $t = 0$ is a special case. The utility function u_0 is a function of one variable defined on E_1^+. It is also assumed to satisfy properties (1), (2) and (3) above.

A sequence $(\mathbf{x}_t) = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$, where

$$\mathbf{x}_0 = (x_0^1, 0, 0, \ldots) \quad \text{and} \quad \mathbf{0} \leq \mathbf{x}_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, 0, \ldots) \quad \text{for} \quad t \geq 1,$$

is said to be an **allocation** whenever $x_{t-1}^t + x_t^t = \omega_{t-1}^t + \omega_t^t$ holds for all $t = 1, 2, \ldots$ (or equivalently, whenever $\sum_{t=0}^{\infty} \mathbf{x}_t = \sum_{t=0}^{\infty} \omega_t$).

A **price** in the overlapping generations model is any sequence $\mathbf{p} = (p^1, p^2, \ldots)$, where $p^t \in E_t'$ for eact t—the vector p^t should be interpreted as representing the prevailing prices at period t.

Definition 3.1. *A non-zero price* $\mathbf{p} = (p^1, p^2, \ldots)$ *is said to* **support an allocation** $(\mathbf{x}_t)$ *whenever*

a) $x \succeq_0 x_0^1$ *in* E_1^+ *implies* $p^1 \cdot x \geq p^1 \cdot x_0^1$*; and*
b) $(x, y) \succeq_t (x_t^t, x_t^{t+1})$ *in* $E_t^+ \times E_{t+1}^+$ *implies* $p^t \cdot x + p^{t+1} \cdot y \geq p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1}$ *for all* $t \geq 1$.

It should be noted that if a price $\mathbf{p} = (p^1, p^2, \ldots)$ supports a given allocation $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$, then $\mathbf{x} \succeq_t \mathbf{x}_t$ implies $\mathbf{p} \cdot \mathbf{x} \geq \mathbf{p} \cdot \mathbf{x}_t$ for all $t = 0, 1, 2, \ldots$. The definition of a Walrasian equilibrium in the overlapping generations model is as follows.

Definition 3.2. *An allocation* $(\mathbf{x}_t)$ *is said to be a* **Walrasian equilibrium** *for the overlapping generations model whenever it can be supported by a non-zero price* $\mathbf{p} = (p^1, p^2, \ldots)$ *such that*

a) $p^1 \cdot x_0^1 = p^1 \cdot \omega_0^1$*; and*
b) $p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1} = p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1}$ *for* $t \geq 1$.

As mentioned above, each Riesz space E_t is assumed equipped with a locally convex-solid topology consistent with the Riesz dual system $\langle E_t, E_t' \rangle$. We shall denote by τ the product topology of the product Riesz space $E_1 \times E_2 \times \cdots$. If $\mathbf{E}$ denotes the ideal of $E_1 \times E_2 \times \cdots$ consisting of all sequences that vanish eventually, i.e., if

$$\mathbf{E} = \{(x_1, x_2, \ldots) \in E_1 \times E_2 \times \cdots : \ \exists \ k \text{ such that } x_i = 0 \text{ for all } i > k\},$$

then it turns out that the topological dual of $(\mathbf{E}, \tau)$ is precisely the product Riesz space

$$\mathbf{E}' = E_1' \times E_2' \times \cdots .$$

The details are included in our next result—whose straightforward proof is omitted.

Theorem 3.3. *The topological dual of* $(\mathbf{E}, \tau)$ *is*

$$\mathbf{E}' = E_1' \times E_2' \times \cdots ,$$

where the duality between $\mathbf{E}$ *and* $\mathbf{E}'$ *is given by*

$$\mathbf{p} \cdot \mathbf{x} = \sum_{t=1}^{\infty} p^t \cdot x_t ,$$

for all $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbf{E}$ *and all* $\mathbf{p} = (p^1, p^2, \ldots) \in \mathbf{E}'$.

It should be immediate from the above discussion that the OLG model defines a pure exchange economy with a countable number of agents $\{0,1,2,\ldots\}$ whose commodity price duality is described by the Riesz dual system $\langle \mathrm{E}, \mathrm{E}' \rangle$. This exchange economy will be referred to simply as the OLG economy.

An important case for the overlapping generations model is when the commodity space at each period is the ideal generated by the total endowment of that period. The total endowment present at period t is given by the vector

$$\theta_t = \omega_{t-1}^t + \omega_t^t \ .$$

The ideal generated by θ_t in E_t will be denoted by Θ_t. That is,

$$\Theta_t = \{\, x \in E_t\colon \text{ There exists } \lambda > 0 \text{ with } |x| \leq \lambda\theta_t \,\}\,.$$

The ideal Θ_t under the norm $\|x\|_\infty = \inf\{\,\lambda > 0\colon\ |x| \leq \lambda\theta_t\,\}$ is an M-space with unit. As usual, the norm dual of $(\Theta_t, \|\cdot\|_\infty)$ will be denoted by Θ_t'. We shall need to consider the overlapping generations model when the commodity price duality at each period is given by the Riesz dual pair $\langle \Theta_t, \Theta_t' \rangle$. Thus, a price for this OLG model is any sequence of the form

$$\mathbf{p} = (p^1, p^2, \ldots)\,,$$

where $0 \leq p^t \in \Theta_t'$ for each t.

We shall denote by $\boldsymbol{\Theta}$ the ideal of $\Theta_1 \times \Theta_2 \times \cdots$ consisting of all sequences having a finite number of non-zero coordinates, i.e.,

$$\boldsymbol{\Theta} = \{(x_1, x_2, \ldots) \in \Theta_1 \times \Theta_2 \times \cdots\colon\ \exists\ k \text{ such that } x_i = 0 \text{ for all } i > k\}\,.$$

By Theorem 3.3, we know that if each Θ_t is equipped with the $\|\cdot\|_\infty$-norm and τ denotes the product topology of $\Theta_1 \times \Theta_2 \times \cdots$, then the topological dual of $(\boldsymbol{\Theta}, \tau)$ is the Riesz space

$$\boldsymbol{\Theta}' = \Theta_1' \times \Theta_2' \times \cdots$$

and the duality of the Riesz dual system $\langle \boldsymbol{\Theta}, \boldsymbol{\Theta}' \rangle$ is given by

$$\mathbf{p} \cdot \mathbf{x} = \sum_{t=1}^{\infty} p^t \cdot x_t\,,$$

for all $\mathbf{x} = (x_1, x_2, \ldots) \in \boldsymbol{\Theta}$ and all $\mathbf{p} = (p^1, p^2, \ldots) \in \boldsymbol{\Theta}'$.

4. A CORE EQUIVALENCE THEOREM

In this section we shall study decentralization properties of allocations for the OLG model. A coalition S is any non-empty set of consumers and may be a finite or an infinite set. We start with a definition.

Definition 4.1. *An allocation* $(\mathbf{x}_t)$ *is said to be:*

1) **Short-term optimal**, *whenever there is no allocation* $(\mathbf{y}_t)$ *satisfying*
   a) $\sum_{t=0}^{\infty} \mathbf{y}_t = \sum_{t=0}^{\infty} \omega_t$;
   b) $\mathbf{y}_t = \mathbf{x}_t$ *for all but a finite number of* t;
   c) $\mathbf{y}_t \succsim_t \mathbf{x}_t$ *for all* t; *and*
   d) $\mathbf{y}_t \succ_t \mathbf{x}_t$ *for at least one* t.
2) *An allocation* $(\mathbf{x}_t)$ *is said to be in the* **short-term core**, *whenever there is no coalition* S *and no assignment* $\{\mathbf{y}_t\colon t \in S\}$ *of consumption bundles satisfying*
   a) $\sum_{t \in S} \mathbf{y}_t = \sum_{t \in S} \omega_t$;
   b) $\mathbf{y}_t = \mathbf{x}_t$ *for all but a finite number of* $t \in S$;
   c) $\mathbf{y}_t \succsim_t \mathbf{x}_t$ *for all* $t \in S$; *and*
   d) $\mathbf{y}_t \succ_t \mathbf{x}_t$ *for at least one* t *in* S.

The notion of short-term optimality is referred to as weak Pareto optimality by Y. Balasko and K. Shell in [9]. In their work Y. Balasko and K. Shell presented interesting necessary and sufficient conditions for an allocation to be short-term optimal; see [9] and [5] for a complete discussion of the concept of short-term optimality. Before studying short-term core allocations, we need to consider the replica of an economy with commodity-price duality $\langle E_t, E_t' \rangle$. The n-**fold replica** OLG economy is a new economy having a countable number of agents $\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, n\}$ such that

1) each agent (i, k) is of type i, i.e., the agent (i, k) has the same characteristics as agent i in the original OLG economy; and
2) The commodity-price duality of this new OLG economy is described by the Riesz dual system $\langle \mathbf{E}, \mathbf{E}' \rangle$.

We now come to the main objective of this section; namely, to state and prove a core equivalence theorem in the overlapping generations model. We shall first derive this result when the overlapping generations model has the Riesz dual system $\langle \mathbf{\Theta}, \mathbf{\Theta}' \rangle$. This theorem will then be generalized in the next section.

Theorem 4.2. *An allocation for the overlapping generations model is a Walrasian equilibrium with respect to the Riesz dual system* $\langle \mathbf{\Theta}, \mathbf{\Theta}' \rangle$ *if and only if it belongs to the short-term core of every n-fold replica of the OLG economy.*

The proof of this theorem will be accomplished by a series of lemmas. For the proof we shall construct a sequence of prices that support the given allocation "locally." A limit of such a sequence will allow us to obtain a price that supports the allocation.

Throughout the proof we shall let $(\mathbf{x}_t)$ be a fixed allocation that belongs to the short-term core of every n-fold replica of the overlapping generations economy. We need to show that $(\mathbf{x}_t)$ is a Walrasian equilibrium. Our first goal is to establish that this allocation admits sequences of prices that support a finite number of consumers.

Lemma 4.3. *Let $(\mathbf{x}_t)$ be an allocation that belongs to the short-term core of every replica of the OLG economy. For each positive integer n there exists a price*

$$\mathbf{p} = (p^1, p^2, \ldots, p^n, p^{n+1}, 0, 0, 0, \ldots) \in \mathbf{\Theta}'$$

such that
a) $p^t \in \Theta'_t$ *for each* $1 \le t \le n+1$;
b) $\mathbf{p} \cdot \mathbf{x}_t = \mathbf{p} \cdot \omega_t > 0$ *for each* $1 \le t \le n$; *and*
c) $\mathbf{x} \in \mathbf{\Theta}$ *and* $\mathbf{x} \succeq_t \mathbf{x}_t$ *for some* $0 \le t \le n$ *implies* $\mathbf{p} \cdot \mathbf{x} \ge \mathbf{p} \cdot \omega_t$.

Proof. Let $\mathbf{I} = \mathbf{A_b}$ denote the ideal generated in $\mathbf{E}$ by $\mathbf{b} = \sum_{t=0}^{n+1} \mathbf{x}_t$. If follows that the first $n+2$ coordinates of $\mathbf{b}$ are non-zero and all coordinates greater than $n+2$ are zero. Clearly,

$$\mathbf{I} = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_{n+1} \times \Omega_{n+2} \times \mathbf{0} \times \mathbf{0} \times \ldots,$$

where $\mathbf{0} = \{0\}$ and Ω_{n+2} denotes the ideal generated by x_{n+1}^{n+2} in E_{n+2}.

Let $\omega_t^* = \omega_t$ for $0 \le t \le n$ and $\omega_{n+1}^* = \sum_{t=0}^{n+1} \mathbf{x}_t - \sum_{t=0}^{n} \omega_t$. From $\sum_{t=0}^{\infty} \omega_t = \sum_{t=0}^{\infty} \mathbf{x}_t$, it follows that $0 \le \sum_{t=0}^{n} \omega_t \le \sum_{t=0}^{n+1} \mathbf{x}_t$ and so $\omega_{n+1}^* \ge 0$ holds. Also, for each $0 \le t \le n+1$ consider the non-empty convex sets

$$F_t = \{\mathbf{x} \in \mathbf{I}^+ \colon \mathbf{x} \succeq_t \mathbf{x}_t\} \quad \text{and} \quad G_t = F_t - \omega_t^*.$$

Let G be the convex hull of $\bigcup_{i=0}^{n+1} G_i$. If $K = \operatorname{Int}(\mathbf{I}^+)$ with respect to the $\|\cdot\|_\infty$-norm of $\mathbf{I}$, then K is a non-empty cone of $\mathbf{I}$, and we claim that $G \cap (-K) = \emptyset$. To see this, assume by way of contradiction that $G \cap (-K) \ne \emptyset$. Then there exist $\mathbf{z}_t \in F_t$ and $\lambda_t \ge 0\, (t = 0, 1, \ldots, n+1)$ with $\sum_{t=0}^{n+1} \lambda_t = 1$ such that

$$\sum_{t=0}^{n+1} \lambda_t (\mathbf{z}_t - \omega_t^*) \in -K.$$

Since each λ_t can be approximated by rational numbers, an easy argument shows that there exist positive integers $m_t\, (t = 0, 1, \ldots, n+1)$ and m such that

$$\sum_{t=0}^{n+1} \tfrac{m_t}{m} (\mathbf{z}_t - \omega_t^*) \in -K.$$

It follows that

$$\sum_{t=0}^{n+1} m_t (\mathbf{z}_t - \omega_t^*) \in -K.$$

Therefore, there exists some $0 < \mathbf{f} \in K$ such that

$$\sum_{t=0}^{n+1} m_t(\mathbf{z}_t - \omega_t^*) = -\mathbf{f}\,.$$

Rearranging the terms, gives

$$\sum_{t=0}^{n+1} m_t\omega_t^* = \sum_{t=0}^{n+1} m_t\left(\mathbf{z}_t + \tfrac{1}{(n+2)m_t}\mathbf{f}\right)\,.$$

Clearly, $\mathbf{z}_t \succeq_t \mathbf{x}_t$ and $\mathbf{z}_t + \frac{1}{(n+2)m_t}\mathbf{f} > \mathbf{x}_t$ for $0 \le t \le n+1$. Thus, $\mathbf{z}_t + \frac{1}{(n+2)m_t}\mathbf{f} \succ_t \mathbf{x}_t$ holds for each $0 \le t \le n+1$. Now let $\mathbf{y}_t = \mathbf{z}_t + \frac{1}{(n+2)m_t}\mathbf{f}$ for $0 \le t \le n+1$ and $\mathbf{y}_t = \mathbf{x}_t$ for $t \ge n+2$. Thus, we have

$$\sum_{t=0}^{n+1} m_t\mathbf{y}_t = \sum_{t=0}^{n+1} m_t\omega_t^*. \tag{1}$$

From (1), we see that

$$\begin{aligned}
\sum_{t=0}^{n+1} m_t\mathbf{y}_t + m_{n+1}\Big(\sum_{t=n+2}^{\infty}\mathbf{y}_t\Big) &= \sum_{t=0}^{n} m_t\omega_t + m_{n+1}\Big(\sum_{t=0}^{n+1}\mathbf{x}_t - \sum_{t=0}^{n}\omega_t\Big) + m_{n+1}\Big(\sum_{t=n+2}^{\infty}\mathbf{x}_t\Big)\\
&= \sum_{t=0}^{n} m_t\omega_t + m_{n+1}\Big(\sum_{t=0}^{\infty}\mathbf{x}_t - \sum_{t=0}^{n}\omega_t\Big)\\
&= \sum_{t=0}^{n} m_t\omega_t + m_{n+1}\Big(\sum_{t=0}^{\infty}\omega_t - \sum_{t=0}^{n}\omega_t\Big)\\
&= \sum_{t=0}^{n} m_t\omega_t + m_{n+1}\Big(\sum_{t=n+1}^{\infty}\omega_t\Big)\\
&= \sum_{t=0}^{n+1} m_t\omega_t + m_{n+1}\Big(\sum_{t=n+2}^{\infty}\omega_t\Big)\,.
\end{aligned}$$

That is,

$$\sum_{t=0}^{n+1} m_t\mathbf{y}_t + m_{n+1}\Big(\sum_{t=n+2}^{\infty}\mathbf{y}_t\Big) = \sum_{t=0}^{n+1} m_t\omega_t + m_{n+1}\Big(\sum_{t=n+2}^{\infty}\omega_t\Big).$$

In view of $\mathbf{y}_t = \mathbf{x}_t$ for all but a finite number of t, $\mathbf{y}_t \succeq_t \mathbf{x}_t$ for all t, and $\mathbf{y}_t \succ_t \mathbf{x}_t$ for at least one t, we see that $(\mathbf{x}_t)$ is not in the short-term core of the $\left(\sum_{t=0}^{n+1} m_t\right)$-replica of the OLG economy. However, this is a contradiction and hence, $G \cap (-K) = \emptyset$.

Since G and $-K$ are both non-empty convex sets and $-K$ is $\|\cdot\|_\infty$-open, it follows from the Hahn–Banach theorem—see, for example, [7, Theorem 9.10, p. 136]—that there exists some nonzero price

$$\mathbf{p} = (p^1, p^2, \ldots, p^{n+1}, p^{n+2}, 0, 0, 0, \ldots)$$

with $p^t \in \Theta'_t$ for $1 \le t \le n+1$ and $p^{n+2} \in \Omega'_{n+2}$ such that $g \in G$ implies $p \cdot g \ge 0$.

Now if $\mathbf{x} \in \mathbf{I}$ and $\mathbf{x} \succeq_t \mathbf{x}_t$ for $0 \le t \le n+1$, then $\mathbf{x} - \omega^*_t \in G$ and so $\mathbf{p}\cdot(\mathbf{x} - \omega^*_t) \ge 0$. That is, $\mathbf{p} \cdot \mathbf{x} \ge \mathbf{p}\cdot\omega^*_t$ holds. Thus, $\mathbf{p} \cdot \mathbf{x}_t \ge \mathbf{p}\cdot\omega^*_t$ holds for $0 \le t \le n+1$. Since $\sum_{t=0}^{n+1} \omega^*_t = \sum_{t=0}^{n+1} \mathbf{x}_t$, it follows that $\mathbf{p} \cdot \mathbf{x}_t = \mathbf{p}\cdot\omega^*_t$. Also $\omega_t = \omega^*_t$ for $0 \le t \le n$, and so $\mathbf{p} \cdot \mathbf{x}_t = \mathbf{p}\cdot\omega_t$.

Since the preferences are strictly monotone, it follows that $p^t \gg 0$ and $\mathbf{p}\cdot\omega_t > 0$ for $0 \le t \le n$. Finally, dropping the p^{n+2} term from $\mathbf{p}$, we get that the price $\mathbf{p} = (p^1, p^2, \ldots, p^n, p^{n+1}, 0, 0, 0, \ldots)$ satisfies the stated properties. ■

Our first goal is to establish that there is a sequence of prices $(\mathbf{p}_n)$, where

$$\mathbf{p}_n = (p^1_n, p^2_n, \ldots, p^n_n, p^{n+1}_n, 0, 0, 0, \ldots),$$

such that each price $\mathbf{p}_n$ "supports the first n consumers." Any such a sequence of prices will be referred to as a fundamental sequence of prices.

Definition 4.4. *A* **fundamental sequence of prices** *for an allocation* $(\mathbf{x}_t)$ *is a sequence of prices* $(\mathbf{p}_n)$*, where* $\mathbf{p}_n = (p^1_n, p^2_n, \ldots, p^n_n, p^{n+1}_n, 0, 0, 0, \ldots)$ *such that:*

a) $0 < p^k_n \in \Theta'_k$ *holds for each* k *and each* n*;*
b) $\mathbf{p}_n \cdot \mathbf{x}_t = \mathbf{p}_n \cdot \omega_t$ *for each* $0 \le t \le n$*; and*
c) $\mathbf{x} \in \mathbf{\Theta}^+$ *with* $\mathbf{x} \succeq_t \mathbf{x}_t$ *for some* $0 \le t \le n$ *implies* $\mathbf{p}_n\cdot\mathbf{x} \ge \mathbf{p}_n\cdot\mathbf{x}_t$.

An immediate consequence of Lemma 4.3 is that fundamental sequences of prices always exist.

Lemma 4.5. *Every allocation* $(\mathbf{x}_t)$ *that belongs to the short-term core of every replica of the OLG economy admits a fundamental sequence of prices* $(\mathbf{p}_n)$ *such that*

$$\mathbf{p}_n \cdot \omega_t > 0$$

holds for all n *and all* t *with* $n \ge t$*.*

Proof. Let n be fixed. By Lemma 4.3 there exists a price

$$\mathbf{p}_n = (p^1_n, p^2_n, \ldots, p^n_n, p^{n+1}_n, 0, 0, 0, \ldots)$$

such that $\mathbf{p}_n \cdot \omega_t > 0$ holds for all $0 \le t \le n$ and $\mathbf{x} \succeq_t \mathbf{x}_t$ in $\mathbf{\Theta}^+$ implies $\mathbf{p}_n \cdot \mathbf{x} \ge \mathbf{p}_n \cdot \omega_t$. The sequence $(\mathbf{p}_n)$ satisfies the desired properties. ■

The next lemma presents a growth estimate for a fundamental sequence of prices and is related to Lemma 4.9 of [4]—and is also the analogue of C. A. Wilson's Lemma 3 in [21].

Lemma 4.6. *Let $(\mathbf{p}_n)$ be a fundamental sequence of prices for the allocation and such that $\mathbf{p}_k \cdot \omega_t > 0$ holds for all k and all t with $k \geq t$. Then for each fixed pair of non-negative integers ℓ and m there exists some constant $M > 0$ (depending only upon ℓ and m) such that*

$$0 < \mathbf{p}_k \cdot \omega_\ell \leq M\, \mathbf{p}_k \cdot \omega_m$$

holds for all $k \geq \max\{\ell, m\}$.

Proof. Let $(\mathbf{p}_n)$ be a fundamental sequence of prices satisfying $\mathbf{p}_k \cdot \omega_t > 0$ for $k \geq t$. Let ℓ and m be fixed and suppose by way of contradiction that our claim is not true. That is, assume that $\liminf_{n \geq r} \frac{p_n \cdot \omega_m}{p_n \cdot \omega_\ell} = 0$, where $r = \max\{\ell, m\}$.

Let $\mathcal{N}$ denote the set of all non-negative integers. Put

$$C_1 = \Big\{ i \in \mathcal{N} \colon \liminf_{n \to \infty} \frac{\mathbf{p}_n \cdot \omega_i}{\mathbf{p}_n \cdot \omega_\ell} = 0 \Big\} \quad \text{and} \quad C_2 = \Big\{ i \in \mathcal{N} \colon \liminf_{n \to \infty} \frac{\mathbf{p}_n \cdot \omega_i}{\mathbf{p}_n \cdot \omega_\ell} > 0 \Big\}.$$

Clearly, $\mathcal{N} = C_1 \cup C_2$, $\ell \in C_2$ and $m \in C_1$. Since $\mathcal{N} = C_1 \cup C_2$, there exists two consecutive integers i and j with $j \in C_1$ and $i \in C_2$.

Now the utility function for consumer i is strictly monotone on the two periods that consumer i lives and thus, we obtain $x_i + \omega_j \succ_i x_i$. Since the utility function u_i is continuous, there exists some $0 < \delta < 1$ with $\delta x_i + \omega_j \succ_i x_i$. Therefore, by the supportability of $\mathbf{p}_n$, we see that

$$\delta \mathbf{p}_n \cdot \mathbf{x}_i + \mathbf{p}_n \cdot \omega_j \geq \mathbf{p}_n \cdot \omega_i$$

holds for all sufficiently large n.

Since $\mathbf{p}_n \cdot \mathbf{x}_i = \mathbf{p}_n \cdot \omega_i$ holds holds for all sufficiently large n, it follows that $\mathbf{p}_n \cdot \omega_j \geq (1 - \delta) \mathbf{p}_n \cdot \omega_i$ holds for all sufficiently large n, and hence

$$\frac{\mathbf{p}_n \cdot \omega_j}{\mathbf{p}_n \cdot \omega_\ell} \geq (1 - \delta) \frac{\mathbf{p}_n \cdot \omega_i}{\mathbf{p}_n \cdot \omega_\ell}$$

holds for all sufficiently large n. Consequently,

$$\liminf_{n \to \infty} \frac{\mathbf{p}_n \cdot \omega_j}{\mathbf{p}_n \cdot \omega_\ell} \geq (1 - \delta) \liminf_{n \to \infty} \frac{\mathbf{p}_n \cdot \omega_i}{\mathbf{p}_n \cdot \omega_\ell},$$

which implies $\liminf_{n \to \infty} \frac{\mathbf{p}_n \cdot \omega_i}{\mathbf{p}_n \cdot \omega_\ell} = 0$, contrary to $i \in C_2$. The proof of the lemma is now complete. ■

Now consider a fundamental sequence of prices $(\mathbf{p}_n)$, where

$$\mathbf{p}_n = (p_n^1, p_n^2, \ldots, p_n^n, p_n^{n+1}, 0, 0, 0, \ldots),$$

for an allocation $(\mathbf{x}_t) = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$. The next lemma tells us that for each k the set $\{\mathbf{p}_n \colon n \geq k\}$ is contained in a weak* compact subset of the dual.

Lemma 4.7. *Let $(\mathbf{p}_n)$ be a fundamental sequence of prices for an allocation satisfying $\mathbf{p}_n \cdot \omega_0 = 1$ and $\mathbf{p}_n \cdot \omega_i > 0$ for $n \geq i$. Then for each fixed k the set of prices $\{p_n^k \colon n \geq k\}$ is a relatively weak* compact subset of Θ'_k.*

Proof. Taking into account that $\mathbf{p}_n \cdot \omega_0 = 1$ for each n, we see by Lemma 4.6 that

$$0 < \mathbf{p}_n \cdot \omega_k \leq M$$

holds for each pair (n, k) with $n \geq k$. In particular, for $n \geq k$ the set of prices $\{p_n^k \colon n \geq k\}$ forms a norm bounded subset of Θ'_k, and thus it is a relatively weak* compact subset of Θ'_k. ∎

We are now ready to complete the proof that the given allocation is a Walrasian equilibrium.

Lemma 4.8. *If an allocation belongs to the short-term core of every replica of the OLG economy, then the allocation is a Walrasian equilibrium with respect to the Riesz dual system* $\langle \Theta, \Theta' \rangle$.

Proof. Let $(\mathbf{x}_t)$ be an allocation that belongs to the short-term core of every replica OLG economy. By Lemma 4.5, there exists a fundamental sequence $(\mathbf{p}_n)$ of prices for $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ such that $\mathbf{p}_k \cdot \omega_i > 0$ holds for all $k \geq i$. Replacing each $\mathbf{p}_k$ by $\frac{\mathbf{p}_k}{\mathbf{p}_k \cdot \omega_0}$, we can assume that $\mathbf{p}_k \cdot \omega_0 = 1$ holds for all k.

For each k there exists—by Lemma 4.7—a closed ball $\mathcal{X}_k$ of Θ'_k with center at zero such that $\{p_n^k \colon n \geq k\} \subseteq \mathcal{X}_k$. The closed ball $\mathcal{X}_k$ equipped with the w*-topology of Θ'_k is a compact topological space. By Tychonoff's classical compactness theorem, the product topological space

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \cdots$$

is also a compact topological space. If

$$\mathbf{p}_n = (p_n^1, p_n^2, \ldots, p_n^n, p_n^{n+1}, 0, 0, 0, \ldots),$$

then note that $\mathbf{p}_n \in \mathcal{X}$ holds for each n. Let $\mathbf{p} = (p^1, p^2, \ldots)$ be an accumulation point of the sequence $(\mathbf{p}_n)$ in $\mathcal{X}$—it should be noticed that each p^k is a w*-accumulation point of the sequence $\{p_n^k \colon n \geq k\}$. Clearly, $\mathbf{p}$ defines a positive linear functional on Θ, i.e., $\mathbf{p} \in \Theta'$.

Now applying Lemma 4.6 with $\ell = 1$, we see that for each fixed m the sequence of real numbers $\{\mathbf{p}_k \cdot \omega_m \colon k = 1, 2, \ldots\}$ is bounded away from zero, and so it follows that $\mathbf{p} \cdot \omega_m > 0$ holds for all m. Now we claim that $\mathbf{p}$ supports the allocation $(\mathbf{x}_t)$ on Θ. To see this, let some $\mathbf{x} \in \Theta^+$ satisfy $\mathbf{x} \succeq_t \mathbf{x}_t$. Note that $n \geq t$ implies

$$\mathbf{p}_n \cdot \mathbf{x} \geq \mathbf{p}_n \cdot \mathbf{x}_t = \mathbf{p}_n \cdot \omega_t\,.$$

It then follows that $\mathbf{p} \cdot \mathbf{x} \geq \mathbf{p} \cdot \mathbf{x}_t = \mathbf{p} \cdot \omega_t$ holds, and therefore $(\mathbf{x}_t)$ is a Walrasian equilibrium that is supported on Θ by the price $\mathbf{p} = (p^1, p^2, \ldots)$. ∎

Finally, we prove the "only if" part of Theorem 4.2.

Lemma 4.9. *If an allocation* $(\mathbf{x}_t)$ *for the overlapping generations model is a Walrasian equilibrium with respect to the Riesz dual system* $\langle \Theta, \Theta' \rangle$, *then it belongs to the short-term core of every* n*-fold replica of the OLG economy.*

Proof. Let $(\mathbf{x}_t)$ be a Walrasian equilibrium supported by price $\mathbf{p} = (p^1, p^2, \ldots)$. We shall only show that $(\mathbf{x}_t)$ is in the short-term core of the economy since the proof that $(\mathbf{x}_t)$ is in the short-term core of every n-fold replica of the economy is identical.

Observe that $\mathbf{p} \cdot \mathbf{x}_t = \mathbf{p} \cdot \omega_t$ holds for all $t \geq 0$. Consequently, we have $p^1 \cdot x_0^1 = p^1 \cdot \omega_0^1$ and $p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1} = p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1}$ for each $t \geq 1$. Since $x_{t-1}^t + x_t^t = \omega_{t-1}^t + \omega_t^t$, it follows by induction that

$$p^t \cdot x_t^t = p^t \cdot \omega_t^t \quad \text{and} \quad p^{t+1} \cdot x_t^{t+1} = p^{t+1} \cdot \omega_t^{t+1} \tag{2}$$

hold for all t.

For each $k \geq 1$, let Q_k be the projection of $E_1 \times E_2 \times E_3 \times \cdots$ onto the first k coordinates, i.e.,

$$Q_k(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_k, 0, 0, \ldots).$$

Note that $Q_k(x_t) = Q_k(\omega_t) = 0$ for $t > k$. Also, by (**2**) it follows that

$$\mathbf{p} \cdot Q_k(x_t) = \mathbf{p} \cdot Q_k(\omega_t) \quad \text{for all } t. \tag{3}$$

Now suppose that $(\mathbf{x}_t)$ is not in the short-term core of the economy. Then, there exist a coalition S and an assignment $\{\mathbf{y}_t : t \in S\}$ such that

a) $\sum_{t \in S} \mathbf{y}_t = \sum_{t \in S} \omega_t$;
b) $\mathbf{y}_t = \mathbf{x}_t$ for all but a finite number of $t \in S$;
c) $\mathbf{y}_t \succeq_t \mathbf{x}_t$ for all $t \in S$; and
d) $\mathbf{y}_t \succ_t \mathbf{x}_t$ for at least one $t \in S$.

Consider the case when the coalition S has a finite number of agents. In this case, it follows from (a) that

$$\sum_{t \in S} \mathbf{p} \cdot \mathbf{y}_t = \mathbf{p} \cdot \Big(\sum_{t \in S} \mathbf{y}_t \Big) = \mathbf{p} \cdot \Big(\sum_{t \in S} \omega_t \Big) = \sum_{t \in S} \mathbf{p} \cdot \omega_t .$$

However, since $\mathbf{y}_t \succ_t \mathbf{x}_t$ holds for some $t \in S$, we must have $\mathbf{p} \cdot \mathbf{y}_t > \mathbf{p} \cdot \mathbf{x}_t$ for some $t \in S$, and so $\sum_{t \in S} \mathbf{p} \cdot \mathbf{y}_t > \sum_{t \in S} \mathbf{p} \cdot \omega_t$ also must hold, which is a contradiction.

Now suppose that the coalition S has an infinite number of agents. In this case, there exists an integer $k \geq 1$ such that $\mathbf{y}_t = \mathbf{x}_t$ for all $t \in S$ greater than or equal to k. Note that $Q_k(\mathbf{y}_t) = \mathbf{0}$ holds for all $t > k$. Let $S_1 = \{t \in S\colon t < k\}$ and $S_2 = \{t \in S\colon t \geq k\}$. Clearly, $S = S_1 \cup S_2$ and using (a) and (b), we see that

$$\sum_{t \in S_2} \mathbf{x}_t + \sum_{t \in S_1} \mathbf{y}_t = \sum_{t \in S_2} \omega_t + \sum_{t \in S_1} \omega_t .$$

Now observe that

$$\begin{aligned}
Q_k(\mathbf{x}_k)+\sum_{t\in S_1}\mathbf{y}_t &= Q_k\Big(\sum_{t\in S_2}\mathbf{x}_t\Big)+Q_k\Big(\sum_{t\in S_1}\mathbf{y}_t\Big)\\
&= Q_k\Big(\sum_{t\in S_2}\mathbf{x}_t+\sum_{t\in S_1}\mathbf{y}_t\Big)\\
&= Q_k\Big(\sum_{t\in S_2}\omega_t+\sum_{t\in S_1}\omega_t\Big)\\
&= Q_k\Big(\sum_{t\in S_2}\omega_t\Big)+Q_k\Big(\sum_{t\in S_1}\omega_t\Big)\\
&= Q_k(\omega_k)+\sum_{t\in S_1}\omega_t\,.
\end{aligned}$$

Applying the price $\mathbf{p}$, yields

$$\mathbf{p}\cdot Q_k(\mathbf{x}_k)+\mathbf{p}\cdot\Big(\sum_{t\in S_1}\mathbf{y}_t\Big)=\mathbf{p}\cdot Q_k(\omega_k)+\mathbf{p}\cdot\Big(\sum_{t\in S_1}\omega_t\Big),$$

which, in view of (**3**), implies that

$$\mathbf{p}\cdot\Big(\sum_{t\in S_1}\mathbf{y}_t\Big)=\mathbf{p}\cdot\Big(\sum_{t\in S_1}\omega_t\Big).$$

But $\mathbf{y}_t \succ_t \mathbf{x}_t$ holds for some $t<k$ and thus, we must have

$$\mathbf{p}\cdot\Big(\sum_{t\in S_1}\mathbf{y}_t\Big)>\mathbf{p}\cdot\Big(\sum_{t\in S_1}\omega_t\Big),$$

which is a contradiction. This contradiction shows that $(\mathbf{x}_t)$ is in the short-term core of the OLG economy. ∎

5. THE PROPER OVERLAPPING GENERATIONS MODEL

The objective of this section is to establish a version of a core equivalence theorem for overlapping generations models with proper preferences by applying the results of the previous section. The commodity-price duality at each period t will be described by a Riesz dual system $\langle E_t, E_t'\rangle$. Then, a price in this overlapping generations model is a sequence $\mathbf{p}=(p^1,p^2,\ldots)$ with each $p^t\in E_t'$. Thus, we will be considering the overlapping generations model having Riesz dual system $\langle \mathbf{E},\mathbf{E}'\rangle$. Notice that $\boldsymbol{\Theta}$ is an ideal of $\mathbf{E}$.

Next we establish a core equivalence theorem for the OLG model having Riesz dual system $\langle \mathbf{E}, \mathbf{E}' \rangle$. To accomplish this, we shall invoke the notion of uniform properness. Proper preferences were introduced by A. Mas-Colell [17]. Let us say that the preference $\succeq_t$ induced by u_t is *uniformly proper* whenever there exist locally convex-solid topologies on E_t and E_{t+1} consistent with the dualities $\langle E_t, E_t' \rangle$ and $\langle E_{t+1}, E_{t+1}' \rangle$ such that each $\succeq_t$ is uniformly proper with respect to the product topology on $E_t \times E_{t+1}$. The preference $\succeq_0$ is uniformly proper whenever it is uniformly proper on E_1. Also, let us say that the overlapping generations model is **proper** whenever

a) Each preference $\succeq_t$ $(t = 0, 1, 2, \ldots)$ is uniformly proper; and
b) The total endowment $\theta_t = \omega_{t-1}^t + \omega_t^t$ present at period t is a strictly positive element of E_t for each $t \geq 1$. (Recall that θ_t is strictly positive whenever $q \cdot \theta_t > 0$ holds for all $0 < q \in E_t'$.)

For proper overlapping generations models a Walrasian equilibrium can be characterized as follows.

Theorem 5.1. *An allocation in a proper overlapping generations model is a Walrasian equilibrium with respect to the Riesz dual system $\langle \mathbf{E}, \mathbf{E}' \rangle$ if and only if it belongs to the short-term core of every n-fold replica of the OLG economy.*

Proof. Let $(\mathbf{x}_t)$ be an allocation that belongs to the short-term core of every replica of the economy. By Theorem 4.2 there exists a price $\mathbf{p} = (p^1, p^2, \ldots) \in \mathbf{\Theta}'$ supporting $(\mathbf{x}_t)$ on $\mathbf{\Theta}$ and satisfying $\mathbf{p} \cdot \omega_t > 0$ for $t = 0, 1, 2, \ldots$. We have $p^t \in \Theta_t'$ for each t.

Now by a Theorem of N. C. Yannelis and W. R. Zame [22] (see also [3, Theorem 9.2]) each linear functional $p^t \colon \Theta_t \longrightarrow \mathcal{R}$ is continuous. Since Θ_t is dense in E_t, it follows that p^t has a continuous extension q^t to all of E_t. Thus, $\mathbf{p}^* = (q^1, q^2, \ldots) \in \mathbf{E}'$, and we claim that $\mathbf{p}^*$ supports $(\mathbf{x}_t)$.

To see this, let $\mathbf{y} \succeq_t \mathbf{x}_t$ in $E_t^+ \times E_{t+1}^+$. Fix $\delta > 0$ and note that $\mathbf{y} + \delta\omega_t \succ_t \mathbf{x}_t$. Now each Θ_i is dense in E_i for a locally convex-solid topology. In particular, the ideal $\Theta_t \times \Theta_{t+1}$ is dense in $E_t \times E_{t+1}$. Thus, there exists a net $\{\mathbf{y}_\alpha\} \subseteq \Theta_t^+ \times \Theta_{t+1}^+$ that converges topologically to $\mathbf{y} + \delta\omega_t$. In view of $\mathbf{y} + \delta\omega_t \succ_t \mathbf{x}_t$ and the continuity of u_t, we can assume that $\mathbf{y}_\alpha \succ_t \mathbf{x}_t$ holds for all α. Thus, by the supportability of $\mathbf{p}$ on $\mathbf{\Theta}$, we get $\mathbf{p} \cdot \mathbf{y}_\alpha \geq \mathbf{p} \cdot \omega_t$ for all α, and by the continuity of $\mathbf{p}^*$ on $E_t \times E_{t+1}$, we see that $\mathbf{p}^* \cdot \mathbf{y} + \delta \mathbf{p}^* \cdot \omega_t \geq \mathbf{p}^* \cdot \omega_t$ for all $\delta > 0$. Therefore, $\mathbf{y} \succeq_t \mathbf{x}_t$ in $E_t^+ \times E_{t+1}^+$ implies $\mathbf{p}^* \cdot \mathbf{y} \geq \mathbf{p}^* \cdot \omega_t$, which shows that the price $\mathbf{p}^* \in \mathbf{E}'$ supports the allocation $(\mathbf{x}_t)$. Therefore, $(\mathbf{x}_t)$ is a Walrasian equilibrium with respect to the Riesz dual system $\langle \mathbf{E}, \mathbf{E}' \rangle$. ■

It should be observed that if each commodity space is finite dimensional, then a continuous utility function is automatically uniformly proper. Thus, the above theorem shows that in a classical overlapping generations model an allocation is a Walrasian equilibrium if and only if it belongs to the short-term core of every n-fold replica of the OLG economy.

Next, we shall show that the set of Walrasian equilibria consists exactly of those allocations which are Walrasian equilibria in the short-term. A t-short-term economy is a finite period economy obtained by truncating the original economy in such a way that a Walrasian equilibrium in the OLG model becomes a Walrasian equilibrium in each truncated economy.

Let $(\mathbf{x}_t)$ be a fixed allocation in the OLG model. Then in the truncated economy, we desire $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t)$ to be an allocation. In the truncated economy, we wish for the last generation to preserve the aggregate endowment. However, it may happen that $\sum_0^t \mathbf{x}_i \neq \sum_0^t \omega_i$. Thus, when the original economy is truncated, the characteristics of the consumer for the last period must be modified. Each agent has the same initial endowments and preferences as before except the last agent has $\sum_0^t \mathbf{x}_i - \sum_0^{t-1} \omega_i$ as his initial endowment.

A *t-short-term economy* is defined to be the t-period economy where each consumer has the same preferences and initial endowments as originally and the characteristics of agent t is modified to have $\sum_0^t \mathbf{x}_i - \sum_0^{t-1} \omega_i$ as his initial endowment. Note that a t-short-term economy is an economy with a finite number of consumers that is constructed in such a way that if $(\mathbf{x}_t)$ is a Walrasian equilibrium in the OLG model, then $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t)$ is a Walrasian equilibrium in the t-short-term economy.

An allocation $(\mathbf{x}_t)$ is said to be a *short-term Walrasian equilibrium* if the allocation $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t)$ is a Walrasian equilibrium for each t-short-term economy. Clearly, any Walrasian equilibrium in the OLG model is a short-term Walrasian equilibrium. Also, a short-term Walrasian equilibrium is in the short-term core of every replica of the economy. This can be easily shown by modifying the proof of Lemma 4.9. These observations when combined with Theorem 5.1 give the following characterization of Walrasian equilibria in the OLG model.

Theorem 5.2. *An allocation in a proper overlapping generations model is a Walrasian equilibrium if and only if it is a short-term Walrasian equilibrium.*

Finally, we remark that our techniques (with some obvious modifications) show that Theorems 4.2 and 5.1 are, in fact, true for the general overlapping generations model. That is, Theorems 4.2 and 5.1 are true for an overlapping generations model where

1) r persons are born in each time period, and
2) each person lives ℓ periods.

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